# Numerical Bifurcation Study of Natural Convection in a Layer of Fluid Subject to Spatially Distributed Heating

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**Abstract.** We present the numerical investigation of Rayleigh-Benard convection (RBC) in a slot whose bottom wall is subject to a long-wavelength heating and the upper wall is isothermal. It is shown that multiple flow structures associated with the same conditions can be produced by changing the history of the heating; this history can be controlled by using different initialization conditions, different continuation strategies in the parameters space as well as by using different numerical solvers. The observed flow structures can be categorized into two generic groups, i.e. symmetric and asymmetric flow structures.

**Keywords.** Thermal Bifurcation, Rayleigh-Benard convection, spatially modulated heating, Spectral Method.
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## 1. INTRODUCTION

Thermal bifurcation which results in multiplicity of solutions for the same problem with exactly the same boundary conditions arises from the nonlinearity of field equations in thermo-fluid problems. To study this phenomenon, many researchers focused on the Rayleigh-Benard Convection (RBC) as a typical example which can occur through several routes and successive bifurcations. In this article, we consider RBC problem with modulated temperature on the lower wall. To the best of our knowledge, few works are available on the present topic. The analytical studies of the modulated RBC problem have been done by Kelly and Pal [1,2] who investigated spatially modulated temperature and spatially periodic geometry. Hossain and Floryan [3] also explored the occurrence of convection due to spatial distribution of heating with fairly small, moderate, and large wave number of heating in fluids with a wide range of Prandtl numbers. In this work, we aim to complete the previous investigations on this subject ([1-3]) by extending analysis toward small and very small heating wave numbers ($0.01< \alpha <0.5$). The bifurcation process as a function of the heating wave number $\alpha$ and the amplitude of the heating $Ra$ have been investigated.

## 2. PROBLEM FORMULATION

Consider a layer of fluid confined between two infinite parallel plates placed apart each other at a distance 2d as shown in Fig.1a. The upper wall is kept at a constant temperature while the lower wall is subjected to a spatially distributed heating which maintains the same mean temperature. Figure 1b shows the temperature distribution on the lower wall presented in terms of $\theta$, where $\theta = T-T_U$, and $T_U$ is the upper wall temperature. The working fluid satisfies the Boussinesq approximation. The spatial pattern of heating is well parameterized by the heating wave number ($\alpha$), the amplitude of the heating is specified in terms of a suitably defined Rayleigh number ($Ra=g\beta T_d d^3/\nu\kappa$), and fluid properties are described in terms of the Prandtl number ($\Pr=\nu/\kappa$). In the above, $g$ is the gravitational acceleration, $\beta$ is the thermal expansion coefficient, $T_d$ is the peak-to-peak amplitude of the lower plate temperature, $\nu$ is the kinematic viscosity, and $\kappa$ is the thermal diffusivity. The temperature field is represented as a sum of the conductive field $\theta_0$ and deviations associated with the convective effects $\theta_1$. We introduce two temperature scales; $T_d$ as the conductive temperature scale and $T_v= T_d \Pr$ as the convective temperature scale. We select $d$ as the length scale, $U_v=\nu/d$ as the velocity scale, and $P_v=\rho U_v^2$ as the pressure scale. The complete dimensionless temperature $\theta$ is scaled using $T_v$, i.e. $\theta(x,y)=\theta_0(x,y)/\Pr+\theta_1(x,y)$. The conductive temperature $\theta_0$ has the form $\theta_0(x,y)=1/4\left[-\sinh(\alpha y)/\sinh(\alpha)+\cosh(\alpha y)/\cosh(\alpha)\right]\cos(\alpha x)$.

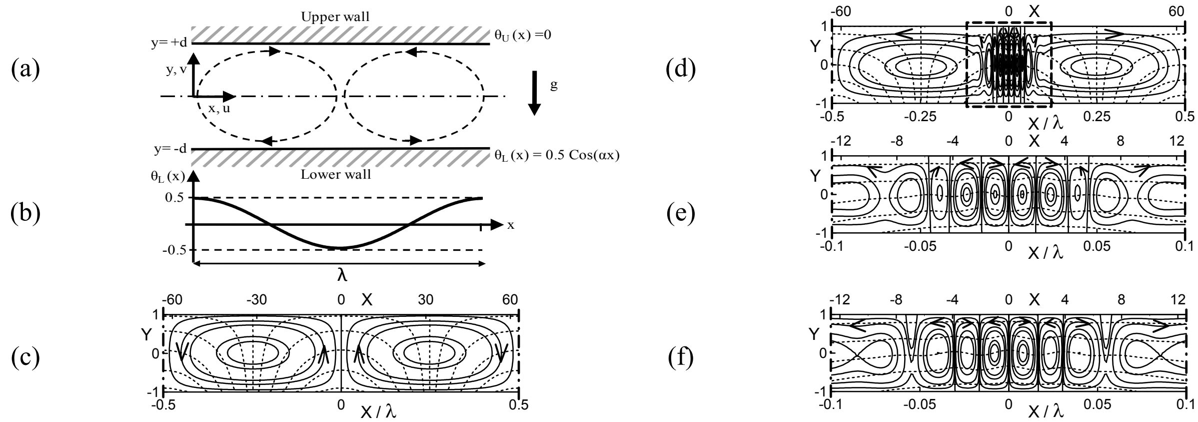


**FIGURE 1.** (a,b) Sketch of the flow system subject to periodic heating imposed at the lower wall – (c-f) Flow structures for heating with Ra=440 and α = 0.05. Fig.1c depicts the primary convection. Fig.1d-f show the secondary convection. Fig.1e displays enlargement of the box from Fig.1d. Solid and dash lines correspond to the streamlines and isotherms, respectively.

The dimensionless field equations have the form (Equations 1.a-d):

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\nabla^2 u\,, \qquad (1.a)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{\partial p}{\partial y}+\nabla^2 v+Ra\theta_1+Ra\,\mathrm{Pr}^{-1}\,\theta_0\,, \qquad (1.b)$$

$$\mathrm{Pr}\frac{\partial \theta_1}{\partial t}+\mathrm{Pr}\left(u\frac{\partial \theta_1}{\partial x}+v\frac{\partial \theta_1}{\partial y}\right)+u\frac{\partial \theta_0}{\partial x}+v\frac{\partial \theta_0}{\partial y}=\nabla^2\theta_1\,, \qquad (1.c)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\,, \qquad (1.d)$$

where $\nabla^2$ denotes the Laplace operator. The boundary conditions take the form $u(\pm 1)=0,\quad v(\pm 1)=0,\quad \theta_1(\pm 1)=0$.

## 3. METHODS OF SOLUTION

In the range of heating wave number that is considered in this work, there is some uncertainty about the accuracy of results; therefore, we have carried out solution using different methods in order to ascertain the validity of the results. Three numerical techniques based on the Spectral Chebyshev-Collocation, the Variable-Step-Size Finite-Difference discretization, and the Finite-Volume Methods have been used to solve the governing equations. Results produced by all these three solvers identify the same characteristics of convection in the zone where multiplicity of solutions exists. Besides, an asymptotic solution has been developed for the limit $\alpha \rightarrow 0$; its testing demonstrated good consistency with the numerical results (for more detail see [4]).

## 4. DISCUSSION OF RESULTS

### 4.1 Symmetric Flow Structure

The primary response of the system appears in the form of symmetric flow structure (with respect to $x$=0) with two counter-rotating convective rolls in each wavelength of heating (Fig.1c). For the case of small heating wave-number α, i.e. long wavelength of heating, small zones on both sides of hot spots are subject to an almost uniform heating. Therefore, if the magnitude of the heating is sufficiently large ($Ra > Ra_{cr}$), the zones around the hot spots may experience the Rayleigh-Benard-type instability. The classical critical Rayleigh number expressed in terms of thickness of the slot for a uniformly heated wall is $Ra_{cr\text{-}classical}$ = 1708 [5]. This number expressed using the present scaling takes the value $Ra_{cr}$ = 427. The numerical results suggest that the thermal instability does take place provided that both $\alpha$ is sufficiently small and $Ra$ exceeds the $Ra_{cr}$ at the same time. In this case, secondary rolls emerge 'only' locally around the hot spots as a result of bifurcation. Two types of secondary convective pattern have been determined: one with the first roll (the roll closest to the hot spot) rotating in the direction opposite to the primary roll as shown in Fig.1e, and the other one with the first roll rotating in direction same as the primary roll (Fig.1f). The other property which makes two types of flow structures completely different is the number of rolls. The one shown in the Fig.1e always has an odd number of roll-pairs while the other one (Fig.1f) always comprises of an even number of roll-pairs. Decrease in the heating wave number leads to nucleation of new rolls, however, both types of structures maintain their characteristics as the new rolls always emerge in pairs.

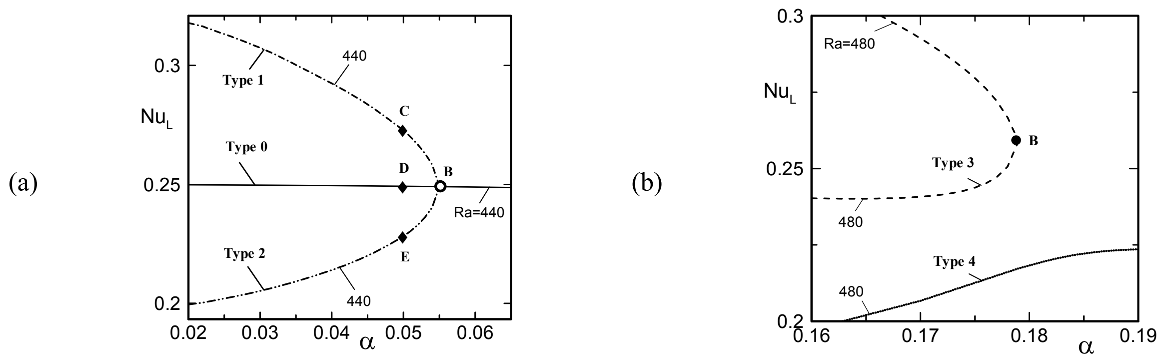


**FIGURE 2** Variations of the local Nusselt number $Nu_L$ as a function of the heating wave number $\alpha$ for a fluid with Pr=0.71 subject to heating corresponding to (a) *Ra*=440 and (b) *Ra*=480. Points C, D, and E in Fig.2a correspond to the same wave number $\alpha$=0.05 but belong to branches of type 1, 0 and 2, with corresponding flow structures displayed in Fig.1e, Fig.1c, and Fig.1f, respectively.

*4.1.1 Pitchfork Bifurcation*

The local Nusselt number at the hot spot was selected as a quantity measuring properties of the flow system for the analysis of the branching process. This number is defined as $Nu_L = -Pr(d\theta/dy)\big|_{x=0,\, y=-1}$. Variation of $Nu_L$ for "supercritical" value of *Ra*=440 is illustrated in Fig.2a. It can be seen that when $\alpha > 0.055$ (point B), the solution is unique. But once the heating wave number decreases below the critical value of 0.055, a pitchfork bifurcation composed of three branches is found. For convenience we shall call solutions corresponding to the middle, upper and lower branches as branches of type 0, 1, and 2, respectively. Branch of type 0 has a simple topology as shown in Fig.1c. For branch of type 1, flow re-arrangement begins with the formation of two small separation bubbles at the upper wall above the hot spot which grow (as α decreases) to form secondary rolls attached to the hot spot. The secondary rolls rotate in direction opposite to primary rolls and thus bring colder fluid into contact with the lower wall resulting in an increase of $Nu_L$. Further decrease of $\alpha$ results in a sequential formation of additional pairs of rolls in both sides of hot spot; this lets the branch of type 1 to always have an odd number of roll-pairs (Fig.1d,e). The same process occurs along the branch of type 2 with one exception, i.e., when we cross the critical point, two secondary roll-pairs (rather than just one) appear to be followed by formation of additional pairs resulting in an even number of roll-pairs (Fig.1f).

*4.1.2 Bifurcation from Infinity*

We shall now focus our attention on higher values of *Ra*, i.e., *Ra* > ~470, where changes in the flow structures correspond to the "bifurcations from infinity" as shown in Fig.2b. For convenience we shall refer to solutions corresponding to the "finite" and "infinite" branches as branches of types 3 and 4, respectively. At the left limit of the lower part of branch of type 3, flow forms one pair of rolls (primary convection). Increase of $\alpha$ results in initiation of the formation of a secondary rolls rotating in the direction opposite to the direction of the primary rolls ($\alpha = 0.177$, point B). The process of formation of new rolls is similar to that observed in the case of branch of type 1 resulting in the creation of an odd number of roll-pairs. At the right limit of branch of type 4, the motion consists of only the primary convection. A decrease of $\alpha$ (moving to the left) leads to formation of secondary rolls ($\alpha = 0.177$). Further decrease in $\alpha$ leads to the process very similar to that observed in the case of branch of type 2.

## 4.2 Asymmetric Structure

So far, we discussed the flow structures preserving left-right symmetry with respect to the plane of *x*=0, but if one adds infinitesimal disturbances to the initial condition the symmetry of the system is broken. Using the combination of purely symmetric and purely asymmetric disturbances with different percentages and different direction of rotation, it is observed that a continuous “fan” of asymmetric branches originates from the same bifurcation point as shown in Fig.3a. To have a better perspective on the asymmetric branches we switch from using the local Nusselt number to using the average Nusselt number over the hot segment of slot (6% of the wavelength) $Nu_T$ as a measure of state of the flow.

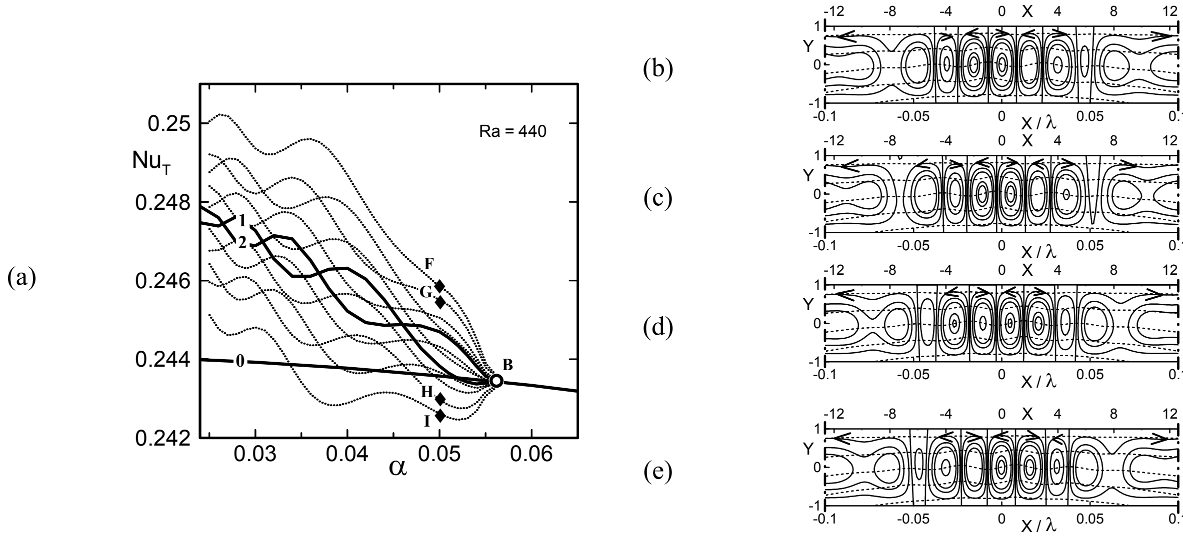


**FIGURE 3** (a)Variations of the average Nusselt number $Nu_T$ as a function of the heating wave number $\alpha$ for a fluid with Pr=0.71 subject to heating corresponding to *Ra*=440. Solid and dotted lines correspond to the symmetric and asymmetric branches, respectively. (b-e) Flow structures for points F-I in Fig.3a, respectively.

Flow structures corresponding to different asymmetric branches illustrate that along each branch the secondary rolls maintain certain configuration with respect to the plane of $x$=0. To distinguish different branches we shall focus on the roll near to $x$=0 (middle roll); the location of the center and the direction of rotation of this roll change from one branch to the other. The two extreme cases obtained using purely asymmetric disturbances (one with clockwise rotation and the other with counterclockwise rotation) are shown in Figs 3b and 3e which correspond to the points F and I in Fig.3a; for these two cases the middle roll is centered at $x$=0. Continuous change in the location of the middle roll results in the formation of continuous bifurcation branches ranging between two purely asymmetric structures (Figs 3b and 3e) and two purely symmetric structures (Figs 2e and 2f). Figures 3c and 3d depict flow structures obtained using combination of 50% purely asymmetric and 50% purely symmetric disturbances and they correspond to points G and H in Fig.3a, respectively.

## 5. CONCLUSION

We have investigated natural convection in an infinite horizontal slot subject to periodic heating with heating wavelength that is large when compared with the slot opening. It has been shown that convection has a simple topology consisting of one pair of counter-rotating rolls per heating period when the Rayleigh number *Ra* does not exceed the critical value of 427. When the heating intensity is larger than the critical value, secondary motions appear which may take either symmetric or asymmetric forms. For the symmetric structures, supercritical pitchfork bifurcations occur only if $427 < Ra < \sim 470$ and $\alpha$ is sufficiently small, i.e., $\alpha < \sim 0.14$. Increase of the heating intensity to $Ra > \sim 470$ results in the secondary motions occurring at larger values of α, i.e. $\alpha > \sim 0.14$, and bifurcation changing character to "bifurcations from infinity". For asymmetric structures, the appearance of a continuous fan of bifurcating branches has been identified with all of them originating from the same bifurcation point.

## 6. REFERENCES


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